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BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED TYPE WITH TWO PARABOLIC LINES

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BOUNDARY-VALUE PROBLEM FOR AN EQUATION OF MIXED TYPE WITH TWO PARABOLIC LINES

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Many works have been devoted to boundary value problems for equations of the mixed type with one parabolic line. Below we consider a boundary value problem for mixed equations with two intersecting lines of a degenerate type. Solution is sought in the semi-plane bounded by the characteristic equation. The problem is reduced to a system of singular integral equations whose solution is obtained by the method developed (for the case of one equation) in reference 1.

We take the following equation for the model equation of a mixed type with /699* two intersecting parabolic lines

$$u_{xx} + \operatorname{sgn}(xy)\cdot u_{yy} = 0. \tag{1}$$

We use Γ_1 and Γ_2 to designate the characteristics $x+y=0$, $x>0$ and $x+y=0$, $y>0$ of equation (1) and let $\Gamma=\Gamma_1 \cup \Gamma_2$. The function ψ assigned on Γ will be taken in the form $\psi_1(x)$ on Γ_1 and in the form $\psi_2(y)$ on Γ_2. The regions $x>0$, $-x<y<0$; $x>0$, $y>0$ and $y>0$, $-y<x<0$ will be designated respectively by $\mathfrak{D}_1^-$, $\mathfrak{D}^+$ and $\mathfrak{D}_2^-$ and the union of these regions by $\mathfrak{D}$. Finally we let $r=\sqrt{x^2+y^2}$.

Problem D. To find the function $u(x, y)$ which is limited for $r\to\infty$ and continuous in $\overline{\mathfrak{D}}$ with derivatives u_x and u_y which are continuous inside $\mathfrak{D}$ (it is assumed that the origin of the coordinates u_x and u_y may have logarithmic singularities), which satisfies equation (1) in the region $\mathfrak{D}$ when $x\neq0$, $y\neq0$ and when the boundary condition is $u|_\Gamma=\psi$.

Theorem. If $\psi_1(t)$ and $\psi_2(t)$ have a derivative in $(0, \infty)$ which satisfies /700 the following expression for any finite values of t

$$|\psi'(t_2) - \psi'(t_1)| < \frac{A}{|\ln|t_2-t_1||\cdot|\ln|\ln|t_2-t_1|||\ldots|\ln|\ln\ldots|\ln|t_2-t_1|\ldots|^p} \tag{2}$$

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($A>0$ and $p>1$ are constants) and $\psi'(t)=0(t^{-q})$, $q>1$ for $t\to\infty$, then the solution of problem D exists and is single-valued; the derivatives u_x and u_y have logarithmic singularity at the origin of the coordinates.

To prove this theorem we introduce the notations $\nu(x) = u_y(x, 0)$, $0 < x < \infty$ and $\mu(y) = u_x(0, y)$, $0 < y < \infty$ and consider the Cauchy-Goursat problem

$$\begin{gathered} u_{xx} - u_{yy} = 0; \quad x > 0, \quad -x < y < 0, \\ u_y(x, 0) = \nu(x), \quad 0 < x < \infty; \quad u|_{\Gamma_1} = \psi_1(x), \quad 0 < x < \infty, \end{gathered} \tag{3}$$

$$\begin{gathered} u_{xx} - u_{yy} = 0; \quad y > 0, \quad -y < x < 0, \\ u_x(0, y) = \mu(y), \quad 0 < y < \infty; \quad u|_{\Gamma_2} = \psi_2(y), \quad 0 < y < \infty \end{gathered} \tag{4}$$

as well as the Neumann problem

$$\begin{gathered} u_{xx} + u_{yy} = 0; \quad x > 0, \quad y > 0, \\ u_y(x, 0) = \nu(x), \quad 0 < x < \infty; \quad u_x(0, y) = \mu(y), \quad 0 < y < \infty \end{gathered} \tag{5}$$

with solution continuous in $\bar{D}+$ and bounded for $r\to\infty$, having a continuous derivative with respect to x when x=0 and a continuous derivative with respect to y when y=0.

Solving problems (3) and (4) we find, respectively, that

$$u(x, y) = -\psi_1(0) + \psi_1\left(\frac{x+y}{2}\right) + \psi_1\left(\frac{x-y}{2}\right) + \int_0^{x+y} \nu(t)\, dt, \tag{6}$$

$$u(x, y) = -\psi_2(0) + \psi_2\left(\frac{y+x}{2}\right) + \psi_2\left(\frac{y-x}{2}\right) + \int_0^{y+x} \mu(t)\, dt, \tag{7}$$

And by solving problem (5) we obtain (assuming that the condition for the Neumann problem to have a solution is satisfied)

$$\begin{aligned} u(x, y) = u_0 &+ \frac{1}{2\pi}\int_0^\infty \nu(t) \ln \frac{[(t+x)^2 + y^2][(t-x)^2 + y^2]}{(x^2 + y^2)^2}\, dt + \\ &+ \frac{1}{2\pi}\int_0^\infty \mu(t) \ln \frac{[(t+y)^2 + x^2][(t-y)^2 + x^2]}{(x^2 + y^2)^2}\, dt, \end{aligned} \tag{8}$$

where $u_0 = \lim_{r\to\infty} u(x, y)$. The conditions $u(x, -0) = u(x, +0)$, $u(-0, y) = u(+0, y)$ /701

now lead to a system of singular integral equations

$$\begin{aligned} \nu(s) + \frac{1}{\pi}\int_0^\infty \frac{2s}{\xi^2 - s^2}\,\nu(\xi)\,d\xi - \frac{1}{\pi}\int_0^\infty \frac{2s}{\xi^2 + s^2}\,\mu(\xi)\,d\xi &= -\psi_1'\left(\frac{s}{2}\right), \\ \mu(s) + \frac{1}{\pi}\int_0^\infty \frac{2s}{\xi^2 - s^2}\,\mu(\xi)\,d\xi - \frac{1}{\pi}\int_0^\infty \frac{2s}{\xi^2 + s^2}\,\nu(\xi)\,d\xi &= -\psi_2'\left(\frac{s}{2}\right). \end{aligned} \tag{9}$$

The method of reducing the total singular integral equations to characteristic equations is applied to system (9). This method is developed (for the case of a single equation) in reference 1. By using this method it is easy to see that in the class of $\mathfrak{H}$ functions which are limited at infinity and which have a logarithmic singularity when s=0, system (9) has a unique solution

$$\begin{aligned} \nu(s) &= -\frac{1}{2}\,\psi_1'\left(\frac{s}{2}\right) + \frac{1}{\pi}\int_0^\infty \left[\frac{\xi}{\xi^2 - s^2}\,\psi_1'\left(\frac{\xi}{2}\right) + \frac{\xi}{\xi^2 + s^2}\,\psi_2'\left(\frac{\xi}{2}\right)\right] d\xi, \\ \mu(s) &= -\frac{1}{2}\,\psi_2'\left(\frac{s}{2}\right) + \frac{1}{\pi}\int_0^\infty \left[\frac{\xi}{\xi^2 - s^2}\,\psi_2'\left(\frac{\xi}{2}\right) + \frac{\xi}{\xi^2 + s^2}\,\psi_1'\left(\frac{\xi}{2}\right)\right] d\xi. \end{aligned}$$

Now it is easy to see that $u_0 = \psi_1(\infty) + \psi_2(\infty) - \psi(0)$. By carrying out a direct verification we convince ourselves that the condition for the Neumann problem to have a solution (5) is satisfied for any ψ_1 and ψ_2 which, in turn satisfy the conditions specified above.

By finding $\nu(x)$ and $\mu(y)$ the solution of problem D is obtained in closed form. The unique value of the solution follows from the equivalence of problem D to system (9) and the unique solution of this system in class $\mathfrak{H}$. In the same way it can be shown that problem D is also correct for the equation $yu_{xx}+xu_{yy}=0$.

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